Optimization of CPM pragmatic capacity

Sergio Benedetto, Guido Montorsi, Alberto Perotti and Alberto Tarable
Center for Multimedia Radio Communications
Politecnico di Torino, Torino, Italy
Email: {sergio.benedetto, guido.montorsi, alberto.perotti, alberto.tarable}@polito.it

Abstract—The paper extends the pragmatic approach to coded continuous-phase modulation (CPM). It first proposes an optimization of the CPM capacity for given complexity and bandwidth efficiency. Then, the best CPM schemes are embedded into a pragmatic coded-modulation scheme, whose pragmatic capacity (a.k.a. BICM capacity) is maximized through a careful design of the mapping between input bits and CPM waveforms. The so obtained CPM schemes show a pragmatic capacity very close to the CPM capacity. An outer binary serially-concatenated convolutional code, cascaded with the CPM modulator through an interleaver, yields performance close to the pragmatic (and so to the CPM) capacity without requiring iterations between outer code and CPM modulator.

I. INTRODUCTION

Continuous phase modulation (CPM) is a class of bandwidth efficient modulation schemes [1] whose characteristic of constant envelope makes them particularly suited to communication systems employing nonlinear amplifiers. Good examples are the second generation GSM cellular system, and satellite communication systems. A CPM modulator is a finite-state machine delivering to the channel a waveform that depends on its input symbol and internal state. In [2] a CPM modulator has been shown to be decomposable into the cascade of a time-invariant convolutional encoder (continuous-phase encoder, CPE) operating on a ring of integers, and of a time-invariant memoryless modulator (MM). Recently, this decomposition has been exploited by inserting an outer convolutional encoder whose coded bits enter an interleaver and then the CPE, thus forming what is known in the literature as a serially-concatenated convolutional encoder (SCCC) [3]. Iterating between the outer encoder and the CPE through the interleaver yields rather good performance [4], [5], [6], which should be compared to the capacity of the CPM scheme. This capacity, in turn, can be evaluated through the techniques explained in [8] or [7]. In the following, we will call this scheme SCCC-CPM.

In communication systems where the channel conditions can vary significantly with time, an efficient radio resource management requires the availability at the physical layer of adaptive coding-modulation, capable of varying its characteristics of bandwidth and energy efficiency following the channel rate of variation. This requirement has originated an active research on bit-interleaved (also known as pragmatic) coded modulation (see [10] and references therein), which consists in cascading a highly performing variable-rate binary encoder (typically, a punctured turbo or low-density parity-check code) with several modulation schemes with increasingly large signal alphabets. An example of the obtainable results has been published in [11], which demonstrates a scheme based on SCCC and linear two-dimensional modulations yielding spectral efficiencies in a very wide range lying around 1 dB from the Shannon capacity limits.

In [12] a first attempt to design pragmatic schemes employing CPM modulation (called P-CPM in the following) has been presented. The authors showed that the pragmatic capacity of CPM schemes heavily depends on the mapping between information bits and CPM signals, and presented a mapping optimization algorithm with some examples. The approach does not require iterations between the outer encoder and the CPE, since the CPM is treated exactly as a linear modulation in a bit-interleaved turbo-trellis coded modulation approach. A nice consequence is that the overall CPE state complexity is not enhanced by the number of iterations, thus permitting to increase the bandwidth efficiency through the use of a larger number of CPE states.

In this paper, first, for a given complexity of the CPM scheme, we optimize the main CPM parameters by maximizing the CPM rate versus the signal-to-noise ratio (SNR). Then, we choose the best CPM schemes for a few rates and complexity, and propose an algorithm to optimize the mapping that stems from the maximization of the pragmatic capacity. Finally, we support the choices with simulation results, showing that great improvements can be obtained with respect to the natural mapping induced by the Rimoldi decomposition. After the optimization, the pragmatic capacity moves significantly closer to the CPM capacity.

II. SYSTEM DESCRIPTION

A CPM modulator is a device with memory that generates continuous-phase, constant envelope modulated waveforms

\[ x(t) = \sqrt{\frac{2E_s}{T}} e^{j\psi(t)} \]  

(1)

whose phase

\[ \psi(t) = 2\pi h \sum_{n=-\infty}^{\infty} a_n q(t - nT) \]  

(2)

depends on the input information symbols \( a_n \in \{ \pm 1, \pm 3, \ldots, \pm (M - 1) \} \), where \( M = 2^m \) is the size of the input alphabet. Here, \( T \) is the symbol interval, \( E_s \) is the energy per symbol, \( h = Q/P \) is the modulation index (Q
and $P$ are relatively prime integers), and $q(t)$ is the phase pulse, a continuous function with the following properties

$$q(t) = \begin{cases} 0 & t \leq 0 \\ \frac{t}{2LT} & 0 < t < LT \\ \frac{L-T}{2} & t \geq LT \end{cases}$$

A CPM scheme is then defined by specifying its parameters $M$, $h$, $L$ and $q(t)$. In this paper, we will only consider REC phase pulses, defined by:

$$q(t) = \frac{t}{2LT}, \quad 0 < t < LT$$

A well known representation of CPM modulators is the Rimoldi decomposition [2]: according to this representation, the modulator is decomposed into the cascade of a continuous phase encoder (CPE) and a memoryless modulator (MM). The CPE is, in general, a time-invariant convolutional encoder operating on a ring of integers.

We are interested in pragmatic coded CPM schemes, represented in their TX and RX part in the block diagram of Fig. 1. By CPM modulator we mean the cascade of CPE and MM, and the CPM SISO demodulator provides the sequence of log-likelihood ratios to the external turbo decoder. No iterations are performed between the external decoder and the CPM demodulator. The channel is additive white Gaussian noise with two-sided power spectral density $N_0/2$.

III. CPM CAPACITY COMPUTATION

With reference to Fig. 1, CPM signals $x$ are infinite-length waveforms. Consider then a finite observation window of $2N+1$ symbol intervals, and define the channel mutual information over it:

$$I(X;Y) = E \left\{ \log_2 \frac{p(y|x)}{p(y)} \right\}$$

where $X = \{X_n\}_{n=-N}^{N}$ is a vector of $(2N+1)$ signals taking values in the set of equally likely $2^m(2N+1)$ CPM waveforms (starting from the zero state) and $Y$ is the corresponding vector of $(2N+1)$ channel outputs. The CPM capacity can then be defined through the limit

$$C_{CPM} = \lim_{N \to \infty} \frac{1}{2N+1} I(X;Y)$$

(Here and in other places, by CPM capacity we mean the constrained capacity with uniform input distribution.)

Using the log-likelihood ratio (LLR) notation $\lambda(u,y) \triangleq \log_2(p(y|u)) - \log_2(p(y|u_{ref}))$ and the max* operator [9]

$$\max^*(a,b) \triangleq \max(a,b) + \log_2(1+2^{-|a-b|})$$

we can transform (3) into

$$C_{CPM} = m - \lim_{N \to \infty} \frac{1}{2N+1} E \left\{ \max_u^* \lambda(u,y) - \lambda(x,y) \right\}.$$  

where $m$ is the number of bits per CPM waveform.

It has been observed in [7] that the first term inside the average is a by-product of the SISO algorithm and corresponds to the max* of the forward path metrics $\alpha$ at step $N$ in the SISO CPM demodulator

$$\max_u^* \lambda(u,y) = \max_s^* \alpha_N(s),$$

with the following initialization

$$\alpha_N(s) = \begin{cases} 0, & s = 0; \\ -\infty, & s \neq 0. \end{cases}$$

Moreover, since the channel is memoryless, the second term is obtained by summing the LLRs of the transmitted waveforms:

$$\lambda(x,y) = \sum_{i=-N}^{N} \lambda(x_i,y_i).$$

Finally, owing to the ergodic properties of the system, the ensemble average can be removed leading to:

$$C_{CPM} = m - \lim_{N \to \infty} \frac{1}{2N+1} E \left\{ \max_u^* \lambda(u,y) - \lambda(x,y) \right\}.$$  

Thus, the CPM capacity can be estimated through a Monte-Carlo simulation of the three internal blocks of Fig. 1, i.e., the CPM modulator, the channel, here assumed to be AWGN, and the CPM soft demodulator, followed by a time average.

A. Optimization procedure

We define the complexity of a CPM scheme as the number of edges per trellis section $C = P^2 + L$ of the CPM trellis, where the parameters have been defined in Section II. Our aim is to maximize the CPM rate measured in bits/sec/Hertz versus the SNR for a given complexity. The optimization algorithm modifies the CPM parameters $(m, P, L)$ yielding the given complexity, and computes the CPM capacity expressed in bits per CPM waveform for a given SNR. Then, based on a bandwidth definition of the CPM schemes, it evaluates the rate and chooses the best scheme for all SNRs of interest. In the following we describe the optimization algorithm step by step:

1) Define the bandwidth $B$ of the CPM system that contains a given percentage of the total power.
2) Define the CPM symbol rate as $R_s \triangleq 1/T$.
3) The symbol SNR $E_s/N_0$ is given by

$$\frac{E_s}{N_0} = \frac{1}{R_s} \frac{P_T}{E_s N_0}.$$
Fig. 2. Spectral efficiency versus $E_b/N_0$ of the best CPM schemes with rectangular pulse and variable complexity. The bandwidth is defined at 99 % of power.

where $P_T$ is the transmitted power.

4) Compute the capacity of each CPM scheme using the method described in Section III as a function of the symbol SNR

$$C_{\text{CPM}} = C_{\text{CPM}}(E_s/N_0).$$

5) Evaluate the spectral efficiency as

$$C = \frac{R_c}{B} C_{\text{CPM}} \left( \frac{1}{R_s} \frac{P_T}{N_0} \right) \text{[bit/s/Hz]}.$$  

6) All CPM schemes with a given complexity $X$ are compared with respect to $C$ for each value of $\frac{P_T}{N_0}$ and the best is chosen.

7) Finally, to obtain the normalized plot that uses the bit SNR $E_b/N_0$ on the abscissa we use the relationship $E_b/N_0 = \frac{1}{C_{\text{CPM}}} \frac{E_s}{N_0} = \frac{R_c}{B} \frac{E_s}{N_0} = \frac{1}{C_{\text{CPM}}} \frac{P_T}{N_0}$.

In Fig. 2 we show the spectral efficiencies versus $E_b/N_0$ for the best CPM schemes with REC frequency pulses and complexity ranging from 4 to 512. The bandwidth is defined at 99% of the power. For comparison purposes, in the figure we have also plotted the spectral efficiencies of QPSK and 8PSK modulations with a square root raised cosine shaping filter with roll-off 0.25.

In Table I we list, for a maximum complexity ranging from 16 to 128, the best CPM schemes achieving the target spectral efficiencies 1 and 1.5 bit/s/Hz, at the minimum value of $E_b/N_0$, the actual spectral efficiency $C$, the CPM parameters $m$, $L$, and $P$, and finally the symbol rate $R_s$.

### IV. PRAGMATIC CAPACITY OF A CPM SCHEME

Now we propose an algorithm that, starting from the capacity-optimized CPM schemes of the previous section, finds the mapping that maximizes the pragmatic capacity.

With reference to Fig. 1, let the sequence $b$ of bits be formed by vectors $B_n = (B_{n,1}, \ldots, B_{n,m})$ of the $m$ bits entering the CPM modulator at the $n$-th trellis step. As previously, we consider a length-$(2N + 1)$ input sequence of $B_n$. To reduce the modulator complexity, we suppose that the mapping between binary inputs and CPM trellis edges is time-invariant.

The optimal mapping between binary input sequences and transmitted signals is the one that maximizes the pragmatic capacity, defined by:

$$C_{\text{CPM}} = \sum_{i=1}^{m} I(B_{0,i}; Y)$$  

for the asymptotic case $N \to +\infty$.

Since $C_{\text{CPM}} = m - \sum_{i=1}^{m} H(B_{0,i}|Y)$, the optimal mapping will minimize the sum of the conditional entropies.

We have:

$$H(B_{0,i}|Y) = \int H(B_{0,i}|Y = y)p_Y(y)dy$$

where $H(B_{0,i}|Y = y)$ is given explicitly by:

$$H(B_{0,i}|Y = y) = E_{b|Y=y} \left( \max_{u} \lambda(u, y) - \max_{u \in X_{0,i}(b)} \lambda(u, y) \right)$$

$X_{0,i}(b)$ being the set of signals corresponding to $B_{0,i} = b$.

To find a viable path to mapping optimization, we approximate the distribution of $Y$ with its value for asymptotically high SNRs. Precisely, in (5) we substitute $p_Y(y) = E_{X}(x - y)$, where we have used Dirac’s delta.

Thus, we have:

$$H(B_{0,i}|Y) \approx E_{X}(E_{b|Y=x}) \left( \max_{u} \lambda(u, x) - \max_{u \in X_{0,i}(b)} \lambda(u, x) \right).$$

Now, we can partition the average w.r.t $x$ according to the value of the $i$-th bit label in the zero section: $E_{X} = E_{b|Y=x_{0,i}(b)}$. By the Jensen’s inequality and the concavity of the entropy function, we obtain the upper bound:

$$H(B_{0,i}|Y) \leq K E_{b|Y=x_{0,i}(b)} \left( \max_{u \in X_{0,i}(b)} \lambda(u, x) - \max_{u \in X_{0,i}(b)} \lambda(u, x) \right).$$

## Table

<table>
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<tr>
<th>$X_{\text{max}}$</th>
<th>C (measured [bit/s/Hz])</th>
<th>$E_b/N_0$ [dB]</th>
<th>$C$ (bits/s/Hz)</th>
<th>$m$</th>
<th>$P$</th>
<th>$L$</th>
<th>$X$</th>
<th>$R_s$</th>
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<td>1</td>
<td>5</td>
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<td>80</td>
<td>1.17</td>
</tr>
</tbody>
</table>

| TABLE I

Table of the best CPM schemes at 99% bandwidth for complexities ranging from 16 to 128.
Owing to the ergodic property of the system, it can be shown that $\max_{\mathbf{x}}^* \lambda(\mathbf{u}, \mathbf{x})$ is a constant for almost every $\mathbf{x}$. Thus:

$$H(B_{0,1} | \mathbf{Y}) \leq K' - K \times$$

$$E_{b0} E_{b|Y \in X_{0,i}(b')} \left( \max_{\mathbf{x} \in X_0,i}^* \max_{\mathbf{b} \in X_0,i}^* \lambda(\mathbf{u}, \mathbf{x}) \right).$$

(9)

Since we want to minimize $\sum_{i=1}^m H(B_{0,1} | \mathbf{Y})$, a viable approach consists in minimizing the sum of the above upper bounds.

Based on (9), our mapping optimization procedure consists in the following steps:

- For every pair of trellis edges, we apply the SISO algorithm by extending the trellis both to the left and to the right, and compute in such a way $\max_{\mathbf{x} \in X_0,i}^* \max_{\mathbf{b} \in X_0,i}^* \lambda(\mathbf{u}, \mathbf{x})$ over all pairs of paths passing through the pair of edges. After a few trellis steps, the path metrics reach a steady-state value, so there is no need to proceed further.

- The previous step yields a metric for each pair of edges. We divide the edges into $M$ clusters, in such a way that the highest metrics are all clustered together. In doing this, we have to take into account the constraints that allow for a right-resolving labelling.

- We map the $M = 2^m$ clusters to $m$-tuples of bits according to a Gray mapping.

The pragmatic capacity of the CPM schemes whose mapping has been optimized using this algorithm will be shown in the next section. It is worth noting that, when the CPM parameters make it possible, the CPE of the optimized scheme results to be non-recursive.

V. NUMERICAL RESULTS

The optimization procedure described in Sec. III-A has been applied to the CPM schemes of Tab. I.

For a rate of 1 bits/s/Hz the best CPM schemes with complexities ranging from 16 to 64 edges per trellis section of Table I have been chosen and the optimization algorithm has been performed. The CPM and P-CPM capacities are reported in Fig. 3. While for CPM capacities the $E_b/N_0$ decreases with increasing complexity (although marginally from 32 to 64), the best pragmatic CPM scheme has a complexity $C = 16$ edges per trellis section. This phenomenon needs to be further investigated. A tentative explanation refers to the CPM parameters. In particular, the distance spectrum of a CPM scheme shows a $P$-fold symmetry. If $M$ divides $P$, this symmetry can be exploited in the binary labelling of the trellis, thus giving a small loss between the CPM capacity and the pragmatic capacity. Instead, if $M$ does not divide $P$ (as for the CPM scheme with complexity 32), this symmetry cannot be fully exploited, and the loss becomes more significant. The intermediate case (with complexity 64) refers to a quaternary scheme, for which the loss of the pragmatic approach w.r.t. binary schemes is likely to increase.

1A labelling of trellis edges is said to be right-resolving if edges leaving the same state have different labels.

Fig. 4 shows the CPM capacity, pragmatic capacities (both optimized and non optimized) and the achievable information rates for the best scheme with complexity 16. By non optimized pragmatic capacity we mean the capacity of the scheme with the natural mapping induced by the Rimoldi decomposition. In order to evaluate by Monte-Carlo simulation the achievable information rates, an outer block code with rate $R_e / (mC)$ has been used. The resulting coded modulation scheme has been referred to in [12] as P-CPM. The considered outer codes belong to the family of serially concatenated convolutional codes defined in [11]. The chosen information word length is $K = 19$, 200 bits. The results show that a gain of around 2 dB can be achieved through the optimization of the mapping.

For rates around 1.5 bits/s/Hz, the CPM and P-CPM capacities are reported in Fig. 5. In this case, $E_b/N_0$ decreases with increasing complexity for a complexity ranging from 16 to 64 for both capacities. Then, the case with complexity 128, with $P = 5$, shows a large gap between the CPM capacity and the pragmatic capacity. The above explanation still holds.
The best pragmatic CPM scheme has thus a complexity of 64 edges per trellis section, and loses less than 1 dB with respect to CPM. For this scheme, Fig. 6 shows the CPM capacity, pragmatic capacities (both optimized and non-optimized) and the achievable information rates. Also in this case the results show that a gain of around 2 dB can be achieved through the optimization of the mapping.

The bit and frame error rates for the best schemes at the two values of spectral efficiency are represented in Fig. 7. As it can be seen, no error floor is observed in either case.

VI. CONCLUSIONS

We have first proposed an optimization of the CPM capacity for given complexity and bandwidth efficiency. Then, the best CPM schemes obtained so far were embedded into a pragmatic coded-modulation scheme, whose mapping between input bits and CPM waveforms has been optimized. The so obtained CPM schemes showed a pragmatic capacity very close to the CPM capacity. An outer binary serially-concatenated convolutional code, cascaded with the CPM modulator through an interleaver, has been shown by simulation to yield performance close to the pragmatic (and so to the CPM) capacity without requiring iterations between outer code and CPM modulator.

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REFERENCES